Ex(1): $x\dfrac{dy}{dx}+(1-x)y=xe^{x}$

$$\frac{dy}{dx}+\frac{1-x}{x}y=e^{x} \Rightarrow P=e^{\int \frac{1-x}{x}dx}=e^{\ln x-x}=xe^{-x}$$

$$y=\frac{1}{xe^{-x}}\int (xe^{-x})(e^{x})\,dx+\frac{c}{xe^{-x}}$$

$$y=\left(\frac{x}{2}+\frac{c}{x}\right)e^{x}$$

H.E.: $y'+y=\sin x \Rightarrow P=e^{\int dx}=e^{x}$

$$\frac{1}{e^{x}}\int \sin x\, e^{x}dx=\frac{1}{e^{x}}\cdot e^{x}(\sin x-\cos x)=\sin x-\cos x$$

Ex(2): $\dfrac{d^3y}{dx^3}-2\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}+2y=0$

$y=e^{rx}$ After substitution of $y=e^{rx}$ in DE

$r^3-2r^2-r+2=0$ $\quad r_1=1,\ r_2=-1,\ r_3=2$ (Distinct roots)

$$y=C_1e^{x}+C_2e^{-x}+C_3e^{2x}$$

Ex(3): $\dfrac{d^3y}{dx^3}-5\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+9y=0$

$r^3-5r^2+3r+9=0$

$r_1=r_2=3,\ r_3=-1$ (repeated)

$$y=c_1e^{3x}+c_2xe^{3x}+c_3e^{-x}$$

---

if $a\mp ib$ is a double root $\quad y=e^{ax}[(c_1+c_2x)\cos bx+(c_3+c_4x)\sin bx]$

Ex(4): $y''+2y'+5y=0$

$y=e^{rx}\Rightarrow r^2+2r+5=0$

$r_{1,2}=-1\mp 2i$

$$y=e^{-x}(c_1\cos 2x+c_2\sin 2x)$$

$y(x) = y_h(x) + y_p(x)$

y_h $\quad y''' - y' = 0$

$y = e^{rx} \Rightarrow r^3 - r = 0 \quad (r^2-1)r = 0$

$r_1 = 0$

$r_{2,3} = \mp 1$

$y_h = C_1 + C_2 e^x + C_3 e^{-x}$

y_p

$h(x)$	Family
$2x+1$	$x, 1 \quad \cdot x = x^2, x$
$-4\cos x$	$\cos x, \sin x$
$2e^x$	$e^x \quad \cdot x = xe^x$

$y_p = Ax^2 + Bx + C\sin x + D\cos x + E xe^x$

$y = C_1 + C_2 e^x + C_3 e^{-x} + Ax^2 + Bx + C\sin x + D\cos x + E xe^x$

$y' = C_2 e^x - C_3 e^{-x} + 2Ax + B + C\cos x - D\sin x + Ee^x + Exe^x$

$y'' = C_2 e^x + C_3 e^{-x} + 2A - C\sin x - D\cos x + Ee^x + Ee^x + Exe^x$

$y''' = C_2 e^x - C_3 e^{-x} - C\cos x + D\sin x + 3Ee^x + Exe^x$

$-2Ax - B - 2C\cos x + 2D\sin x + 2Ee^x = 2x + 1 - 4\cos x + 2e^x$

$-2A = 2 \quad A = -1$

$-B = 1 \quad B = -1$

$-2C = -4 \quad C = 2$

$2D = 0 \quad D = 0$

$2E = 2 \quad E = 1$

$y_p = -x^2 - x + 2\sin x + xe^x$

$y = y_h + y_p = C_1 + C_2 e^x + C_3 e^{-x} - x^2 - x + 2\sin x + xe^x$

Ex(6): $y'' - 2y' + 2y = e^x \sin x$

y_h $y = e^{rx}$ $\quad r^2 - 2r + 2 = 0$ $\quad y_h = e^x(C_1 \cos x + C_2 \sin x)$

$r_{1,2} = 1 \pm i$

y_p

$h(x)$	Family
$e^x \sin x$	$e^x \sin x, e^x \cos x$. $\xrightarrow{\cdot x} x e^x \sin x, x e^x \cos x$

$y_p = A x e^x \sin x + B x e^x \cos x$

$y_p' = A e^x \sin x + A x e^x \sin x + A x e^x \cos x + B e^x \cos x + B x e^x \cos x - B x e^x \sin x$

$y_p'' = A e^x \cos x + A e^x \sin x + A e^x \sin x + A x e^x \sin x + A x e^x \cos x + A e^x \cos x + A x e^x \cos x$
$+ A x e^x \sin x + B e^x \cos x - B e^x \sin x + B e^x \cos x + B x e^x \cos x - B x e^x \sin x$
$- B e^x \sin x - B x e^x \sin x - B x e^x \cos x$

$A = 0$ $\quad y = e^x(C_1 \cos x + C_2 \sin x) - \frac{1}{2} x e^x \cos x$

$B = -1/2$

Ex(7): $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = 6x + 12$

$x = e^z$

$$\left.\begin{aligned} x\frac{dy}{dx} &= \frac{d}{dz} y = \frac{dy}{dz} \\ x^2 \frac{d^2y}{dx^2} &= \frac{d}{dz}\left(\frac{d}{dz} - 1\right) y = \frac{d^2y}{dz^2} - \frac{dy}{dz} \end{aligned}\right\} \text{subst in DE.}$$

Ex(10)

Ex:(8) $x^2y''+5xy'+4y=0$ $\quad y(x)=?$

$y=x^r \Rightarrow xy'=rx^r \quad x^2y''=r(r-1)x^r$

$[r(r-1)+5r+4]x^r=0$

$r^2-r+5r+4=0$

$r^2+4r+4=0$

$r_{1,2}=-2$

$y=C_1x^{-2}+C_2\log(x)x^{-2}=(C_1+C_2\log x)x^{-2}$

Ex(9) $y''-2y'+2y=e^x\tan x$ $\quad \Rightarrow h(x)=e^x\tan x$, $a_0(x)=1$

$y=e^{rx}$

y_h: $r^2-2r+2=0 \quad r_{1,2}=1\mp i$

$y_h(x)=C_1e^x\cos x+C_2e^x\sin x$

$C_1'u_1+C_2'u_2=0$

$C_1'u_1'+C_2'u_2'=h(x)/a_0(x)$

$y_p(x)=C_1(x)e^x\cos x+C_2(x)e^x\sin x$

$C_1'e^x\cos x+C_2'e^x\sin x=0$

$C_1'e^x(\cos x-\sin x)+C_2'e^x(\sin x+\cos x)=e^x\tan x$

$W=\begin{vmatrix}\cos x & \sin x\\ \cos x-\sin x & \sin x+\cos x\end{vmatrix}=1$

$C_1'=\begin{vmatrix}0 & \sin x\\ \tan x & \sin x+\cos x\end{vmatrix}=-\sin x\tan x$

$C_1=-\int\sin x\tan x\,dx+d_1, \quad C_2=\int\sin x\,dx+d_2$

$y_p(x)=\left[-\int\sin x\tan x\,dx+d_1\right]e^x\cos x+\left[\int\sin x\,dx+d_2\right]e^x\sin x$

$y(x)=y_g=\left(-\int\sin x\tan x\,dx\right)e^x\cos x+d_1e^x\cos x+\left(\int\sin x\,dx\right)e^x\sin x+d_2e^x\sin x$

y_h

y_p

Ex(10): $x^2y''-4xy'+6y=x^3$; $y=x^3$ is one sol.

$y=v(x)x^3$

$y'=v'x^3+3x^2v$

$y''=v''x^3+3x^2v'+3x^2v'+6xv=v''x^3+6x^2v'+6xv$

$x^2(v''x^3+6x^2v'+6xv)-4x(v'x^3+3x^2v)+6vx^3$

$v(\cancel{6x^3-12x^3+6x^3})+x^5v''+2x^4v'=x^3$

0

$x^2v''+2xv'=1$

$v''+\frac{2}{x}v'=\frac{1}{x^2}$

let $v'=w$ $\quad w'+\frac{2}{x}w=\frac{1}{x^2}$

$P=e^{\int\frac{2}{x}dx}=e^{2\ln x}=x^2$

$v'=w=\frac{1}{x}+\frac{C_1}{x^2}$

$y=\frac{1}{x^2}\int \cancel{x^2}\cdot\frac{1}{\cancel{x^2}}dx+\frac{C}{x^2}=\frac{1}{x}dx$

$v=\int\left(\frac{1}{x}+\frac{C}{x^2}\right)dx$

$\frac{1}{x^4}$

$v=\ln x-\frac{C_1}{x}+C_2$

$y=(\ln x-\frac{C}{x}+C_2)x^3=x^3\ln x-C_1x^2+C_2x^3$

Ex(11): $-1+(x-2)+(x-2)^2+\frac{8}{10}(x-2)^3+\frac{11}{17}(x-2)^4+\frac{14}{26}(x-2)^5+\cdots=\sum_{n=0}^{\infty}\frac{3n-1}{n^2+1}(x-2)^n$

$\lim_{n\to\infty}\left|\frac{3n+2}{(n+1)^2+1}\Big/\frac{3n-1}{n^2+1}\right|=\frac{\infty}{\infty}\Rightarrow A_0+A_1(x-x_0)+\cdots$

$A_{n+1}=\frac{3(n+1)-1}{(n-1)^2+1}$

Apply L'hopital rule

at $x=1$;

$y=A_0+A_1(x-x_0)+A_2(x-x_0)^2+\cdots$

$y=A_0+A_1(1-2)+A_2(1-2)^2+\cdots$

$=A_0-A_1+A_2-\cdots$

Converges

at $x=3$;

$y=A_0+A_1+A_2+A_3-\cdots$

diverge

$[1,3)$

Ex(12): $y'' + xy' + xy' - y = 0$ $x_0 = 0$ [we assume that the soln. in the form $y = \sum_{n=0} A_n (x - x_0)^n$]

$$y = \sum_{n=0}^{\infty} A_n x^n, \quad y' = \sum_{n=0}^{\infty} n A_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) A_n x^{n-2}$$

$$\sum_{n=0}^{\infty} n(n-1) A_n x^{n-2} + \underbrace{x \sum_{n=0}^{\infty} n A_n x^{n-1} - \sum_{n=0}^{\infty} A_n x^n}_{\sum_{n=0}^{\infty} (n-1) A_n x^n} = 0$$

$$\sum_{n=0}^{\infty} n(n-1) A_n x^{n-2} + \sum_{n=2}^{\infty} (n-3) A_{n-2} x^{n-2} = 0$$

$$\sum_{n=2}^{\infty} \left(n(n-1) A_n + (n-3) A_{n-2} \right) x^{n-2} = 0$$

$$A_n = -\frac{n-3}{n(n-1)} A_{n-2}, \quad n = 2, 3, ---$$

$n=2$ $A_2 = \frac{1}{2} A_0$

$n=3$ $A_3 = -\frac{0}{6} = 0$

$n=4$ $A_4 = -\frac{1}{12} A_2 = -\frac{1}{24} A_0$

$n=5$ $A_5 = 0$

$n=6$ $A_6 = \frac{1}{240} A_0$

$$y(x) = A_0 + A_1 x + A_2 x^2 + ---$$

$$= A_0 + A_1 x + \frac{1}{2} A_0 x^2 + 0 - \frac{1}{24} A_0 x^4 + --$$

$$= A_0 \left(1 + \frac{1}{2} - \frac{1}{24} + \frac{1}{240} + --- \right) + A_1 x$$

Ex(13)

Ex(12): $y''+xy'+xy'-y=0$ $x_0=0$ [we assume that the soln. in the form $y=\sum_{n=0} A_n(x-x_0)^n$]

$$y=\sum_{n=0}^{\infty} A_n x^n,\ y'=\sum_{n=0}^{\infty} nA_n x^{n-1},\ y''=\sum_{n=0}^{\infty} n(n-1)A_n x^{n-2}$$

$$\sum_{n=0}^{\infty} n(n-1)A_n x^{n-2} \underbrace{+x\sum_{n=0}^{\infty} nA_n x^{n-1} - \sum_{n=0}^{\infty} A_n x^n}_{\sum_{n=0}^{\infty}(n-1)A_n x^n} = 0$$

$$\sum_{n=0}^{\infty} n(n-1)A_n x^{n-2} + \sum_{n=2}^{\infty}(n-3)A_{n-2}x^{n-2}=0$$

$$\sum_{n=2}^{\infty}\left(n(n-1)A_n+(n-3)A_{n-2}\right)x^{n-2}=0$$

$$A_n=-\frac{n-3}{n(n-1)}A_{n-2},\quad n=2,3,\ldots$$

$n=2$ $A_2=\frac{1}{2}A_0$

$n=3$ $A_3=-\frac{0}{6}=0$

$n=4$ $A_4=-\frac{1}{12}A_2=-\frac{1}{24}A_0$

$n=5$ $A_5=0$

$n=6$ $A_6=\frac{1}{240}A_0$

$$y(x)=A_0+A_1x+A_2x^2+\ldots$$

$$=A_0+A_1x+\frac{1}{2}A_0x^2+0-\frac{1}{24}A_0x^4+\ldots$$

$$=A_0\left(1+\frac{1}{2}-\frac{1}{24}+\frac{1}{240}+\ldots\right)+A_1x$$

Ex(13)

$(2s^2 - s) A_0 = 0 \Rightarrow 2s(s-1) = 0 \equiv f(s)$ indicial eqn.
$\quad A_0 \neq 0 \qquad s = 0, \; s = +\frac{1}{2}$

$A_n = \frac{(2s+\cancel{2n-1})}{(n+s)(2s+\cancel{2n-1})} A_{n-1} \quad n \geq 1$

$A_n = \frac{1}{n+s} A_{n-1} \quad n \geq 1$

$s_i \quad 0, \; \frac{1}{2}$

$s_1 = 0, \; s_2 = \frac{1}{2} \qquad s_1 - s_2 = -\frac{1}{2} \neq$ integer
two linearly solutions exists

$\boxed{s = s_1 = 0} \quad A_n = \frac{2(n+1)+1}{n(2n-1)} A_{n-1} \quad n \geq 1 \qquad y = \sum A_n x^{n+\cancel{s}}$ 0

$A_1 = A_0$
$A_2 = \frac{1}{2} A_1 = \frac{1}{2} A_0$
$A_3 = \frac{1}{3} A_0$
$A_4 = \frac{1}{12} A_0$
$\vdots$

$y_1(x)\; A_0 + A_1 x + A_2 x^2 + \cdots$
$y_1 = A_0 + A_0 x + \frac{A_0}{2} x^2 + \frac{A_0}{6} x^3 + \cdots$
$y_1 = A_0 (1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots)$
$\Rightarrow A_0 e^x = y_1(x)$

$s = s_2 = \frac{1}{2} \qquad A_n = \frac{2(n-\frac{1}{2})+1}{(n+\frac{1}{2})2n} A_{n-1} \quad n \geq 1$

$A_1 = \frac{2}{3} A_0$
$A_2 = \frac{4}{15} A_0$
$A_3 = \frac{8}{3 \times 5 \times 7} A_0$
$\vdots$

$y_2(x) = \sum_{n=0}^{\infty} A_n x^{\frac{1}{2}+n} = x^{\frac{1}{2}} [A_0 + A_1 x + A_2 x^2 + \cdots]$
$= x^{1/2} [A_0 + \frac{2}{3} A_0 + \cdots] = A_0 x^{1/2} [1 + \frac{2}{3} x + \frac{8}{3 \times 5 \times 7} x^2 \cdot]$

$y(x) = y_1(x) + y_2(x)$
$= A_0 e^x + A_1 x^{1/2} [1 + \frac{2}{3} x + \cdots]$

Ex (15). $x^2y'' + (4-x)y' + (2-x)y = 0$

$P(x) = \frac{4-x}{x}$, $Q(x) = \frac{2-x}{x^2}$ } at $x=0$ singular

$x \cdot \frac{4-x}{x}$, $x^2 \cdot \frac{2-x}{x^2}$ } at $x=0$ regular.

a $x=0$ regular singular point so apply Frobenius Method.

$y = \sum_{n=0}^{\infty} A_n x^{n+s}$ $\quad y' = \sum_{n=0}^{\infty} (n+s) A_n x^{(n+s-1)}$ $\quad y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) A_n x^{(n+s-2)}$

$x^2 \sum_{n=0}^{\infty} (n+s)(n+s-1) A_n x^{n+s-2} + (4-x) \sum_{n=0}^{\infty} (n+s) A_n x^{(n+s-1)} + (2-x) \sum_{n=0}^{\infty} A_n x^{n+s}$

$\sum_{n=0}^{\infty} (n+s)(n+s-1) A_n x^{n+s} + \sum_{n=0}^{\infty} 4(n+s) A_n x^{(n+s)} - \sum_{n=0}^{\infty} (n+s) A_n x^{n+s+1} +$

$\sum_{n=0}^{\infty} 2 A_n x^{n+s} - \sum_{n=0}^{\infty} A_n x^{n+s+1}$

$\sum_{n=0}^{\infty} ((n+s)(n+s-1) + 4(n+s) + 2) A_n x^{n+s} - \sum_{n=1}^{\infty} (n+s\cancel{-1+1}) A_{n-1} x^{n+s}$

$(s(s-1) + 4s + 2) A_0 x^s + \sum \underbrace{([(n+s)(n+s-1) + 4(n+s) + 2] A_n + (n+s) A_{n-1})}_{=0} x^{n+s}$

$-1 \quad -2$

$s_1 - s_2 > 0$

$-1 - (-2) = 1$

$A_n = \frac{(s+n) A_{n-1}}{(s+n)(s+n+3) + 2}$

$A_1 = 0$

$A_2 = 0$

$y = x^{-1} \sum A_n x^n$

$\boxed{y(x) = A_0 x^{-1}}$ first linearly independent solution.

For the linearly independent soln.

$$y_2(x) = C u(x) \ln x + x^{S_2} \sum_{n=0}^{\infty} B_n x^n \quad , \quad B_n = \left[\frac{d}{ds} (s - S_2) A_n(s) \right] \Bigg|_{s=S_2}$$

$$A_n = \frac{s+n}{(s+n)(s+n+3)+2} * A_{n-1}$$

$$y_1(x) = A_0 u(x)$$

$$y_2(x) = C x^{-1} \ln x + x^{-2} \sum_{n=0}^{\infty} B_n x^n$$

$$B_0 = \left[\frac{d}{ds} (s+2) \; A_0(s) \right]_{s=-2} = \frac{d}{ds} (s+2) A_0 \Big|_{s=-2} = A_0$$

$$B_1 = \left[\frac{d}{ds} (s+2) A_1(s) \right]_{s=-2} = \frac{d}{ds} \left(\frac{(s+2)\;(s+1)\;A_0}{(s+1)(s+4)+2} \right) \Bigg|_{s=-2} \qquad A_1(s) = \frac{s+1}{(s+1)(s+4)+2} A_0$$

$$= \frac{d}{ds} \left(\frac{s+1}{s+3} A_0 \right) \Bigg|_{s=-2} = 2A_0$$

$$B_2 = \left[\frac{d}{ds} (s+2) \; A_2(s) \right]_{s=-2} = \frac{d}{ds} \left(\frac{(s+2)(s+2)}{(s+2)(s+5)+2} \; A_1(s) \right) \Bigg|_{-2} = -\frac{1}{2} A_0$$

$$B_3 = -\frac{1}{12} A_0$$

$$\vdots$$

$$B_n = -\frac{1}{(n-1)!\,n!} A_0$$

$$y_2(x) = C x^{-1} \ln x + x^{-2} \left[B_0 + B_1 x + B_2 x^2 + B_3 x^3 - - - \right]$$

$$= C x \ln x + x^{-2} \left[A_0 + 2A_0 x - \frac{1}{2} A_0 x^2 - - - \right]$$

$$y(x) = A_0 x^{-1}$$

$$y = C x^{-1} \ln x + A_0 \left[x^{-2} + 2x^{-1} + - - - \right] + A_0 x^{-1}$$

For the linearly independent soln.

$$y_2(x) = C\,u(x)\ln x + x^{S_2}\sum_{n=0}^{\infty} B_n x^n \, , \qquad B_n = \left[\frac{d}{ds}(s-s_2)\,A_n(s)\right]\Bigg|_{s=s_2}$$

$$A_n = \frac{s+n}{(s+n)(s+n+3)+2} * A_{n-1}$$

$$y_1(x) = A_0\, u(x)$$

$$y_2(x) = C x^{-1}\ln x + x^{-2}\sum_{n=0}^{\infty} B_n x^n$$

$$B_0 = \left[\frac{d}{ds}(s+2)\;A_0(s)\right]_{s=-2} = \frac{d}{ds}(s+2)A_0\Big|_{s=-2} = A_0$$

$$B_1 = \left[\frac{d}{ds}(s+2)\,A_1(s)\right]_{s=-2} = \frac{d}{ds}\left(\frac{(s+2)\,(s+1)\,A_0}{(s+1)(s+4)+2}\right)\Bigg|_{s=-2} \qquad A_1(s) = \frac{s+1}{(s+1)(s+4)+2}A_0$$

$$= \frac{d}{ds}\left(\frac{s+1\,A_0}{s+3}\right)\Bigg|_{s=-2} = 2A_0$$

$$B_2 = \left[\frac{d}{ds}(s+2)\;A_2(s)\right]_{s=-2} = \frac{d}{ds}\left(\frac{(s+2)(s+2)}{(s+2)(s+5)+2}\;A_1(s)\right)\Bigg|_{-2} = -\frac{1}{2}A_0$$

$$B_3 = -\frac{1}{12}A_0$$

$$\vdots$$

$$B_n = -\frac{1}{(n-1)!\,n!}\;A_0$$

$$y_2(x) = Cx^{-1}\ln x + x^{-2}\left[B_0 + B_1 x + B_2 x^2 + B_3 x^3 - - -\right]$$

$$= C x \ln x + x^{-2}\left[A_0 + 2A_0 x - \frac{1}{2}A_0 x^2 - - -\right]$$

$$y(x) = A_0 x^{-1}$$

$$y = C x^{-1}\ln x + A_0\left[x^{-2} + 2x^{-1} + - - -\right] + A_0 x^{-1}$$

Ex (12): $u'' + x u' - u = 1 + x^2$ $x = 0$ ordinary point

$$y = \sum_{n=0} A_n x^n , \quad y' = \sum_{n=0} n A_n x^{n-1} , \quad y'' = \sum_{n=0} n(n-1) A_n x^{n-2}$$

$$\sum_{n=0} n(n-1) A_n x^{n-2} + x \sum_{n=0} n A_n x^{n-1} - \sum_{n=0} A_n x^n$$

$$\sum_{n=2}^{\infty} \left(n(n-1) A_n + (n-3) A_{n-2} \right) x^{n-2} = 1 + x^2$$

$$(2 A_2 - A_0) x^0 + (6 A_3) x + 12 A_4 + A_2 x^2 + \sum_{n=5}^{\infty} G_n x^{n-3} = 1 + x^2$$

$$2 A_2 - A_0 = 1 \qquad 6 A_3 = 0 \qquad 12 A_4 + A_2 = 1$$

$$A_2 = \frac{1 + A_0}{2} \qquad A_3 = 0 \qquad A_4 = \frac{1 - A_2}{12} = \frac{1}{24}(1 - A_0)$$

$$G_n = 0; \quad A_n = -\frac{n-3}{n(n-1)} A_{n-2} \quad n \geq 5$$

$A_5 = 0$

$A_6 = -\frac{3}{30} A_4 = -\frac{1}{10} \cdot \frac{1}{24} (1 - A_0)$

$y(x) = A_0 + A_1 x + A_2 x^2 + \text{———}$

$\quad = A_0 + A_1 x + \left(\frac{1 + A_0}{2}\right) x^2 + 0 + \frac{1}{12} (1 - A_0) x^4 + \dots$

$\quad = A_0 \left[1 + \frac{1}{2} x^2 - \frac{1}{24} x^4 + \dots \right] + A_1 [x + \dots] + \left(\frac{1}{2} x^2 + \frac{1}{24} x^4 + \dots \right)$

$$A_n = \frac{-[(n-2)(n-6)}{(n-4)(n+4)+16}$$

$\sum_{n=0}^{\infty} B_n x^n$ where $B_n = \frac{d}{ds}\left(A_n(s)\right)\Big|_{s=-4}$

$$B_0 = \frac{d}{ds} A_0(s)\Big|_{s=-4} = A_0$$

$$B_1 = \frac{d}{ds}\left(A_1 s\right) = 0$$

$$B_2 = \frac{d}{ds}\left(A_2(s)\right)\Big|_{s=-4} = \frac{d}{ds}\left(-\frac{s(s+4)+4}{(s+2)(s+6)+16}\right)\Big|_{s=-4} = \frac{d}{ds}\left(\frac{-(s+2)^2}{(s+6)^2}\right)_{s=-6}$$

$$\frac{-2(s+2)(s+6)^2 + 2(s+6)(s+2)^2}{(s+6)^4}\Bigg|_{s=-4} = \frac{-2(-2)(2)^2 + 2(2)(-2)^2}{2^4} = 2A_0$$

$$y_2(x) = C_1 x^{-4}(1-x^2)\ln x + x^{-4}(1+2A_0x^2)$$

$$y(x) = y_1(x) + y_2(x) = C_1 x^{-4}(1-x^2)\ln x + C_2\left(x^{-4} + x^{-2}\right)$$